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Eight Step Block Modified Backward Differentiation Formulae for the Treatment of Stiff Initial Value Problems of Ordinary Differential Equations

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Abstract— ***The backward differentiation formulae are considered as the most popular and the best class of linear multistep methods for handling stiff problems, however, they have their drawbacks as they are A – stable for $k = 1, 2, A(\alpha)$, stable for $k = 3, \dots, 6$ and non self – starting for $k \geq 2$. This paper focused on the development and implementation of a new higher order numerical method by a modification of the backward differentiation formulae for the solutions of stiff initial value problems. The methodology used for the derivation of the new method was the multistep collocation approach, while the test equation approach was used to plot their regions of absolute stability. From the convergence analysis, the methods were consistent and zero – stable, hence convergent and of uniform order $p = k$. Also, results show that the eight step BMBDF were A- stable. The new method resolves the stability problem and overcomes the non self - starting property inherent in the standard backward differentiation formulae (BDF). The eight step BMBDF was applied to odes arising from real life and results show that they were efficient and accurate and compete well with other existing methods. Also, the solution curves of the eight step BMBDF compete well with the exact solutions and the well-known ode23 solver. Since the new method was A – stable, it was recommended for the solutions of stiff initial value problems resulting from real life.***

Keywords — Backward differentiation, A – stability, region of absolute stability, convergence, consistency.

I. INTRODUCTION

Differential equations, which describe how quantities change across time or space, arise naturally in science and engineering, and indeed in almost every field of study where measurements are taken [8]. A wide variety of these natural phenomena arising in the physical world are modelled into ordinary differential equations (ODEs). Unfortunately, many of these problems do not have analytic solutions, the Robertson and Vanderpol equations are examples, hence the need for good numerical methods to approximate their solutions.

Many researchers have proposed various forms of linear multistep methods for the solutions of stiff ordinary differential equations, among which is the Backward Differentiation Formula (BDF). The BDF is considered as the most popular class of linear multistep methods which was introduced by [6]. These methods can be efficiently used for the solutions of stiff problems, the only drawback being the lack of A-Stability for order exceeding two. Since their introduction, most of the improvements in the class of linear multistep methods have been based on them because of their special properties [3].

A lot of extensions and improvements have been made on the basis of the backward differentiation formulae such as; fully implicit 3- point block backward differentiation formula (BBDF) developed by [11], three – step optimized block backward differentiation formulae (TOBBDF) developed by [15], L – Stable Block Backward Differentiation Formula developed by [2] and Extended Block Backward Differentiation formula (EBBDF) developed by [9]. [12] developed the Extended 3 – point Superclass of Block Backward Differentiation Formula for solving stiff initial value problems. [1] developed the Improved Two-Point Block Backward Differentiation Formulae for Solving First Order Stiff Initial Value Problems of Ordinary Differential Equations. The methods were derived by modifying the existing Two-Point Block Backward Differentiation Formulae for Solving First Order Stiff Initial Value Problems of Ordinary Differential Equations (12BBDF) of [10]. Convergence and stability analysis established that the new methods were A – stable. [4] developed a Two-step Hybrid Block Backward Differentiation Formula for the solution of stiff ordinary differential equations

This paper uses the approach similar to that of [5] by modifying the Backward Differentiation Formulae and hence constructing new block methods for the solutions of stiff problems of ordinary differential equations using multistep collocation approach of [13]

II. THE BACKWARD DIFFERENTIATION FORMULAE

The backward differentiation formula, a numerical method in the category of linear multistep method falls in the family of implicit methods for the numerical integration of ordinary differential equations.

The general formula is given as

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\beta_k f_{n+k} \qquad (1)$$

III. THE MODIFIED BACKWARD DIFFERENTIATION FORMULAE

The modified backward differentiation formulae are a new class of methods derived by modifying the backward differentiation formulae (BDF) defined in (1).

The general k step MBDF is defined by the difference equation

$$\sum_{j=0}^{k-2}\alpha_j y_{n+j} + \alpha_k y_{n+k} = h\left[\beta_{v-2} f_{n+v-2} + \beta_{v-1} f_{n+v-1}\right] \qquad (2)$$

where

$$v = \frac{k}{2} \text{ for even } k \text{ and } \frac{k+1}{2} \text{ for odd } k \qquad (3)$$

IV. DERIVATION OF THE MODIFIED BACKWARD DIFFERENTIATION FORMULAE

The modified backward differentiation formulae (MBDF) is derived using the multistep collocation approach of Sirisena [13] and Onumanyi *et al* [14]. The procedure involves the construction of numerical methods of step number $k = 8$

The continuous formulation of the k- step MBDF (2) is given as

$$y(x) = \sum_{j=0}^{k-2}\alpha_j(x) y_{n+j} + h\left[\beta_{v-2}(x) f_{n+v-2} + \beta_{v-1}(x) f_{n+v-1}\right] \qquad (4)$$

where v is as given in (3). Applying the multistep approach,

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & . & . & . & x_n^k \\ 1 & x_{n+1} & x_{n+1}^2 & . & . & . & x_{n+1}^k \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & x_{n+k-2} & x_{n+k-2}^2 & . & . & . & x_{n+k-2}^k \\ 0 & 1 & 2x_{n+v-2} & . & . & . & kx_{n+v-2}^{k-1} \\ 0 & 1 & 2x_{n+v-1} & . & . & . & kx_{n+v-1}^{k-1} \end{bmatrix} \qquad (5)$$

and the elements of $C = D^{-1}$ are given in (6)

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & . & . & . & \alpha_{k-2,1} & h\beta_{v-2,1} & h\beta_{v-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & . & . & . & \alpha_{k-2,2} & h\beta_{v-2,2} & h\beta_{v-1,2} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \alpha_{0,k+1} & \alpha_{1,k+1} & . & . & . & \alpha_{k-2,k+1} & h\beta_{v-2,k+1} & h\beta_{v-1,k+1} \end{bmatrix} \qquad (6)$$

D and C must satisfy

$$DC = I \qquad (7)$$

From (7) it follows that the columns of $C = D^{-1}$ give the continuous coefficients $\alpha_j(x), j = 0, 1, \ldots, k-2$, $h\beta_{v-1}(x)$ and $h\beta_{v-2}(x)$.

V. EIGHT STEP BLOCK MODIFIED BACKWARD DIFFERENTIATION FORMULAE (BMBDF)

Substituting $k = 8, v = 4$ in (4) gives,

$$y(x) = \sum_{j=0}^{6}\alpha_j(x) y_{n+j} + h\left[\beta_2(x) f_{n+2} + \beta_3(x) f_{n+3}\right] \qquad (8)$$

Applying the multistep collocation approach, its continuous formulation is

$$\begin{aligned} y(\eta + x_n) = {} & \frac{(6h-\eta)(h-\eta)(4h-\eta)(5h-\eta)(3h-\eta)^2(2h-\eta)^2}{4320h^8} y_n + \frac{\eta(5h-\eta)(6h-\eta)(4h-\eta)(3h-\eta)^2(2h-\eta)^2}{240h^8} y_{n+1} \\ & + \frac{\eta(h-\eta)(4h-\eta)(5h-\eta)(6h-\eta)(26h-19\eta)(3h-\eta)^2}{576h^8} y_{n+2} - \frac{\eta(5h-\eta)(h-\eta)(6h-\eta)(4h-\eta)^2(2h-\eta)^2}{36h^8} y_{n+3} \\ & - \frac{\eta(6h-\eta)(5h-\eta)(h-\eta)(3h-\eta)^2(2h-\eta)^2}{96h^8} y_{n+4} + \frac{\eta(h-\eta)(4h-\eta)(6h-\eta)(3h-\eta)^2(2h-\eta)^2}{720h^8} y_{n+5} \\ & - \frac{\eta(4h-\eta)(5h-\eta)(h-\eta)(3h-\eta)^2(2h-\eta)^2}{8640h^8} y_{n+6} + \frac{\eta(5h-\eta)(2h-\eta)(4h-\eta)(h-\eta)(6h-\eta)(3h-\eta)^2}{48h^7} f_{n+2} \\ & + \frac{\eta(h-\eta)(3h-\eta)(4h-\eta)(5h-\eta)(6h-\eta)(2h-\eta)^2}{36h^7} f_{n+3} \end{aligned} \qquad (9)$$

Evaluating (9) at $\eta = 7h, 8h$ and its first derivative at $\eta = h, 4h, 5h, 6h, 7h, 8h$ gives the BMBDF method for $k = 8$.

$$y_{n+7} + 70y_{n+5} - 350y_{n+4} - 525y_{n+3} + 749y_{n+2} + 70y_{n+1} - \frac{10}{3}y_n = -420hf_{n+2} - 700hf_{n+3} \ldots\ldots (a)$$

$$y_{n+8} - 70y_{n+6} + 560y_{n+5} - 3150y_{n+4} - 5376y_{n+3} + 7350y_{n+2} + 720y_{n+1} - 35y_n = -4200hf_{n+2} - 6720hf_{n+3} \ldots\ldots (b)$$

$$-\frac{1}{501}y_{n+6} + \frac{5}{167}y_{n+5} - \frac{50}{167}y_{n+4} - \frac{300}{167}y_{n+3} + \frac{175}{167}y_{n+2} + y_{n+1} + \frac{10}{501}y_n = -\frac{60}{167}hf_{n+1} - \frac{300}{167}hf_{n+2} - \frac{200}{167}hf_{n+3} \ldots\ldots (c)$$

$$\frac{1}{375}y_{n+6} - \frac{8}{125}y_{n+5} - y_{n+4} + y_{n+2} + \frac{8}{125}y_{n+1} - \frac{1}{375}y_n = -\frac{12}{25}hf_{n+2} - \frac{32}{25}hf_{n+3} - \frac{12}{25}hf_{n+4} \ldots\ldots (d)$$

$$-\frac{1}{60}y_{n+6} - \frac{127}{300}y_{n+5} + \frac{3}{2}y_{n+4} + y_{n+3} - \frac{23}{12}y_{n+2} - \frac{3}{20}y_{n+1} + \frac{1}{150}y_n = hf_{n+2} + 2hf_{n+3} - \frac{1}{5}hf_{n+5} \ldots\ldots (e)$$

$$\frac{91}{1350}y_{n+6} - \frac{4}{15}y_{n+5} + y_{n+4} + \frac{32}{27}y_{n+3} - \frac{11}{6}y_{n+2} - \frac{4}{25}y_{n+1} + \frac{1}{135}y_n = hf_{n+2} + \frac{16}{9}hf_{n+3} - \frac{1}{45}hf_{n+6} \ldots\ldots (f)$$

$$-\frac{143}{1068}y_{n+6} + y_{n+5} - \frac{2845}{534}y_{n+4} - \frac{755}{89}y_{n+3} + \frac{4225}{356}y_{n+2} + \frac{302}{267}y_{n+1} - \frac{29}{534}y_n = -\frac{597}{89}hf_{n+2} - \frac{1955}{178}hf_{n+3} - \frac{1}{178}hf_{n+7} \ldots (g)$$

$$y_{n+6} - \frac{11768}{1331}y_{n+5} + \frac{69345}{1331}y_{n+4} + \frac{124800}{1331}y_{n+3} - \frac{167755}{1331}y_{n+2} - \frac{16776}{1331}y_{n+1} + \frac{823}{1331}y_n = \frac{96660}{1331}hf_{n+2} + \frac{151968}{1331}hf_{n+3} + \frac{12}{1331}hf_{n+8} \ldots (h)$$

(10)

VI. CONVERGENCE AND STABILITY ANALYSIS OF THE EIGHT STEP BLOCK MODIFIED BACKWARD DIFFERENTIATION FORMULAE (BMBDF)

Here, the order and stability region of the Eight Step Block Modified Backward Differentiation Formulae (BMBDF) shall respectively be computed and plotted.

Defn 8.1

The order of the k - step linear multistep method $\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j}$ is p if $c_0 = c_1 = \dots = c_p = 0$ *but* $c_{p+1} \neq 0$

where

$$\vec{c}_0 = \sum_{j=0}^{k}\vec{\alpha}_j$$

$$\vec{c}_1 = \sum_{j=0}^{k}(j\vec{\alpha}_j - \vec{\beta}_j)$$

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$$\vec{c}_q = \frac{1}{q!}\sum_{j=0}^{k}[j^q\vec{\alpha}_j] - \frac{1}{(q-1)!}\sum_{j=0}^{k}[j^{(q-1)}\vec{\beta}_j], q = 2,3,\dots \tag{11}$$

The BMBDF (10) is equivalent to

$$\begin{bmatrix} 70 & 749 & -525 & -350 & 70 & -\frac{35}{3} & 1 & 0 \\ 720 & 7350 & -5376 & -3150 & 560 & -70 & 0 & 1 \\ -501 & -525 & 900 & 150 & -15 & 1 & 0 & 0 \\ -24 & -375 & 0 & 375 & 24 & -1 & 0 & 0 \\ -45 & -575 & 300 & 450 & -127 & -5 & 0 & 0 \\ -216 & -2475 & 1600 & 1350 & -360 & 91 & 0 & 0 \\ -16776 & -167755 & 124800 & 69345 & -11768 & 1331 & 0 & 0 \\ -1208 & -12675 & 9060 & 5690 & -1068 & 143 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \\ y_{n+8} \end{bmatrix} +$$

$$\begin{bmatrix} -\frac{10}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -35 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 823 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 58 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_{n} \\ y_{n-2} \\ y_{n-3} \\ y_{n-4} \\ y_{n-5} \\ y_{n-6} \\ y_{n-7} \\ y_{n-8} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & -420 & -700 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4200 & -6720 & 0 & 0 & 0 & 0 & 0 \\ 180 & 900 & 600 & 180 & 0 & 0 & 0 & 0 \\ 0 & 180 & 480 & 0 & 180 & 0 & 0 & 0 \\ 0 & 300 & 600 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1350 & 2400 & 0 & 0 & 30 & 0 & 0 \\ 0 & 96660 & 151968 & 0 & 0 & 0 & 0 & 12 \\ 0 & 7164 & 11730 & 0 & 0 & 0 & 6 & 0 \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \\ f_{n+8} \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} f_{n} \\ f_{n-2} \\ f_{n-3} \\ f_{n-4} \\ f_{n-5} \\ f_{n-6} \\ f_{n-7} \\ f_{n-8} \end{bmatrix} \tag{12}$$

where,

$$\bar{\alpha}_0 = \begin{pmatrix} -\frac{10}{3} \\ -35 \\ 10 \\ 1 \\ 2 \\ 10 \\ 823 \\ 58 \end{pmatrix}, \bar{\alpha}_1 = \begin{pmatrix} 70 \\ 720 \\ -501 \\ -24 \\ -45 \\ -216 \\ -16776 \\ -1208 \end{pmatrix}, \bar{\alpha}_2 = \begin{pmatrix} 749 \\ 7350 \\ -525 \\ -375 \\ -575 \\ -2475 \\ -167755 \\ -12675 \end{pmatrix}, \bar{\alpha}_3 = \begin{pmatrix} -525 \\ -5376 \\ 900 \\ 0 \\ 300 \\ 1600 \\ 124800 \\ 9060 \end{pmatrix}, \bar{\alpha}_4 = \begin{pmatrix} -350 \\ -3150 \\ 150 \\ 375 \\ 450 \\ 1350 \\ 69345 \\ 5690 \end{pmatrix}, \bar{\alpha}_5 = \begin{pmatrix} 70 \\ 560 \\ -15 \\ 24 \\ -127 \\ -360 \\ -11768 \\ -1068 \end{pmatrix},$$

$$\bar{\alpha}_6 = \begin{pmatrix} -\frac{10}{3} \\ -70 \\ 1 \\ -1 \\ -5 \\ 91 \\ 1331 \\ 143 \end{pmatrix}, \bar{\alpha}_7 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \bar{\alpha}_8 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \bar{\beta}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \bar{\beta}_1 = \begin{pmatrix} 0 \\ 0 \\ 180 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \bar{\beta}_2 = \begin{pmatrix} -420 \\ -4200 \\ 900 \\ 180 \\ 300 \\ 1350 \\ 96660 \\ 7164 \end{pmatrix}, \bar{\beta}_3 = \begin{pmatrix} -700 \\ -6720 \\ 600 \\ 480 \\ 600 \\ 2400 \\ 151968 \\ 11730 \end{pmatrix}, \bar{\beta}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 180 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\bar{\beta}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -60 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \bar{\beta}_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 30 \\ 0 \\ 0 \end{pmatrix}, \bar{\beta}_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 6 \end{pmatrix}, \bar{\beta}_8 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 12 \\ 0 \end{pmatrix} \tag{13}$$

Substituting the values of

$\overrightarrow{\alpha_0}, \overrightarrow{\alpha_1}, \overrightarrow{\alpha_2}, \overrightarrow{\alpha_3}, \overrightarrow{\alpha_4}, \overrightarrow{\alpha_5}, \overrightarrow{\alpha_6}, \overrightarrow{\alpha_7}, \overrightarrow{\alpha_8}, \overline{\beta_0}, \overline{\beta_1}, \overline{\beta_2}, \overline{\beta_3}, \overline{\beta_4}, \overline{\beta_5}, \overline{\beta_6}, \overline{\beta_7}, \overline{\beta_8}$

into (11) gives

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = 0$$

and

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$$c_9 = \begin{pmatrix} 2.78\times10^{-1} \\ 3.33 \\ 1.19\times10^{-1} \\ -4.76\times10^{-2} \\ -1.19\times10^{-1} \\ -7.14\times10^{-1} \\ -8.34 \\ -5.07 \end{pmatrix} \qquad (14)$$

Since $c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = 0$ *but* $c_9 = c_{p+1} \neq 0$, the eight step BMBDF has order $p = 8$ and its error constant is as shown in (14). Thus, the BMBDF $k = 8$ is consistent.

VII. ZERO STABILITY OF THE EIGHT STEP BMBDF

Using the approach of [7], the eight step BMBDF (18) expressed as

$$\begin{bmatrix} 70 & 749 & -525 & -350 & 70 & \frac{-35}{3} & 1 & 0 \\ 720 & 7350 & -5376 & -3150 & 560 & -70 & 0 & 1 \\ -501 & -525 & 900 & 150 & -15 & 1 & 0 & 0 \\ -24 & -375 & 0 & 375 & 24 & -1 & 0 & 0 \\ -45 & -575 & 300 & 450 & -127 & -5 & 0 & 0 \\ -216 & -2475 & 1600 & 1350 & -360 & 91 & 0 & 0 \\ -16776 & -167755 & 124800 & 69345 & -11768 & 1331 & 0 & 0 \\ -1208 & -12675 & 9060 & 5690 & -1068 & 143 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \\ y_{n+8} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 35 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -823 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -58 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_n \\ y_{n-2} \\ y_{n-3} \\ y_{n-4} \\ y_{n-5} \\ y_{n-6} \\ y_{n-7} \\ y_{n-8} \end{bmatrix} +$$

$$\begin{bmatrix} 0 & -420 & -700 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4200 & -6720 & 0 & 0 & 0 & 0 & 0 \\ 180 & 900 & 600 & 180 & 0 & 0 & 0 & 0 \\ 0 & 180 & 480 & 0 & 180 & 0 & 0 & 0 \\ 0 & 300 & 600 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1350 & 2400 & 0 & 0 & 30 & 0 & 0 \\ 0 & 96660 & 151968 & 0 & 0 & 0 & 0 & 12 \\ 0 & 7164 & 11730 & 0 & 0 & 0 & 6 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \\ f_{n+8} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n-2} \\ f_{n-3} \\ f_{n-4} \\ f_{n-5} \\ f_{n-6} \\ f_{n-7} \\ f_{n-8} \end{bmatrix} \qquad (15)$$

where,

$$A^{(0)} = \begin{bmatrix} 70 & 749 & -525 & -350 & 70 & \frac{-35}{3} & 1 & 0 \\ 720 & 7350 & -5376 & -3150 & 560 & -70 & 0 & 1 \\ -501 & -525 & 900 & 150 & -15 & 1 & 0 & 0 \\ -24 & -375 & 0 & 375 & 24 & -1 & 0 & 0 \\ -45 & -575 & 300 & 450 & -127 & -5 & 0 & 0 \\ -216 & -2475 & 1600 & 1350 & -360 & 91 & 0 & 0 \\ -16776 & -167755 & 124800 & 69345 & -11768 & 1331 & 0 & 0 \\ -1208 & -12675 & 9060 & 5690 & -1068 & 143 & 0 & 0 \end{bmatrix},$$

$$A^{(1)} = \begin{bmatrix} \frac{10}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 35 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -823 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -58 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (16)$$

Substituting the values of $A^{(0)}$ and $A^{(1)}$ into the equation $\rho(\lambda) = \det[\lambda A^{(0)} - A^{(1)}] = 0$ and evaluating gives $\lambda = 0, 0, 0, 0, 0, 0, 0, 0, 1$. Hence, the method is zero stable.

VIII. REGION OF ABSOLUTE STABILITY OF THE EIGHT STEP BMBDF

The approach used by [16] is applied to the block methods to determine their regions of absolute stability. Using the test equation $y' = \lambda y$ with $z = \lambda h, f = \lambda y, f_n = \lambda y_n, f_{n+1} = \lambda y_{n+1}, f_{n+2} = \lambda y_{n+2}$ is substituted into (18) to give

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$$y_{n+7}=\frac{35}{3}y_{n+6}-70y_{n+5}+350y_{n+4}+525y_{n+3}-749y_{n+2}-70y_{n+1}+\frac{10}{3}y_n-420h\lambda y_{n+2}-700h\lambda y_{n+3}\ldots\ldots(a)$$

$$y_{n+8}=70y_{n+6}-560y_{n+5}+3150y_{n+4}+5376y_{n+3}-7350y_{n+2}-720y_{n+1}+35y_n-4200h\lambda y_{n+2}-6720h\lambda y_{n+3}\ldots\ldots(b)$$

$$y_{n+1}=\frac{1}{501}y_{n+6}-\frac{5}{167}y_{n+5}-\frac{50}{167}y_{n+4}+\frac{300}{167}y_{n+3}-\frac{175}{167}y_{n+2}-\frac{10}{501}y_n-\frac{60}{167}h\lambda y_{n+1}-\frac{300}{167}h\lambda y_{n+2}-\frac{200}{167}h\lambda y_{n+3}\ldots\ldots(c)$$

$$y_{n+2}=-\frac{1}{375}y_{n+6}+\frac{8}{125}y_{n+5}+y_{n+4}-\frac{8}{125}y_{n+1}+\frac{1}{375}y_n-\frac{12}{25}h\lambda y_{n+2}-\frac{32}{25}h\lambda y_{n+3}-\frac{12}{25}h\lambda y_{n+4}\ldots\ldots(d)$$

$$y_{n+3}=\frac{1}{60}y_{n+6}+\frac{127}{300}y_{n+5}-\frac{3}{2}y_{n+4}+\frac{23}{12}y_{n+2}+\frac{3}{20}y_{n+1}-\frac{1}{150}y_n+h\lambda y_{n+2}+2h\lambda y_{n+3}-\frac{1}{5}h\lambda y_{n+5}\ldots\ldots(e)$$

$$y_{n+4}=-\frac{91}{1350}y_{n+6}+\frac{4}{15}y_{n+5}-\frac{32}{27}y_{n+3}+\frac{11}{6}y_{n+2}+\frac{4}{25}y_{n+1}-\frac{1}{135}y_n+h\lambda y_{n+2}+\frac{16}{9}h\lambda y_{n+3}-\frac{1}{45}h\lambda y_{n+6}\ldots\ldots(f)$$

$$y_{n+5}=\frac{143}{1068}y_{n+6}+\frac{2845}{534}y_{n+4}+\frac{755}{89}y_{n+3}-\frac{4225}{356}y_{n+2}-\frac{302}{267}y_{n+1}+\frac{29}{534}y_n-\frac{597}{89}h\lambda y_{n+2}-\frac{1955}{178}h\lambda y_{n+3}-\frac{1}{178}h\lambda y_{n+7}\ldots\ldots(g)$$

$$y_{n+6}=\frac{11768}{1331}y_{n+5}-\frac{69345}{1331}y_{n+4}-\frac{124800}{1331}y_{n+3}+\frac{167755}{1331}y_{n+2}+\frac{16776}{1331}y_{n+1}-\frac{823}{1331}y_n+\frac{96660}{1331}h\lambda y_{n+2}+\frac{151968}{1331}h\lambda y_{n+3}\ldots\ldots(h)$$
$$+\frac{12}{1331}h\lambda y_{n+8}$$

(17)

Expressing (17) in matrix form gives;

$$\begin{bmatrix} 70 & 420z+749 & 700-525 & -350 & 70 & -\frac{35}{3} & 1 & 0 \\ 720 & 4200z+7350 & 6720z-5376 & -3150 & 560 & -70 & 0 & 1 \\ 1+\frac{60}{167}z & \frac{175}{167}+\frac{300}{167}z & \frac{175}{167}+\frac{300}{167}z & \frac{-50}{167} & \frac{5}{167} & \frac{-1}{501} & 0 & 0 \\ \frac{8}{125} & \frac{12}{25}z+1 & \frac{32}{25}z & \frac{12}{25}z-1 & -\frac{8}{125} & \frac{1}{375} & 0 & 0 \\ -\frac{3}{20} & -z-\frac{23}{12} & -2z+1 & \frac{3}{2} & \frac{1}{5}z-\frac{127}{300} & -\frac{1}{60} & 0 & 0 \\ -\frac{4}{25} & -z-\frac{11}{6} & -\frac{16}{9}z+\frac{32}{17} & 1 & -\frac{4}{15} & \frac{91}{1350}-\frac{1}{45}z & 0 & 0 \\ \frac{302}{267} & \frac{597}{89}z+\frac{4225}{356} & \frac{1955}{178}z-\frac{755}{89} & \frac{-2845}{534} & 1 & \frac{-143}{1068} & \frac{1}{178}z & 0 \\ \frac{-16776}{1331} & -\frac{96660}{1331}z-\frac{167755}{1331} & -\frac{151968}{1331}+\frac{124800}{1331} & \frac{69345}{1331} & -\frac{11768}{1331} & 1 & 0 & \frac{-12}{1331}z \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \\ y_{n+8} \end{bmatrix}$$

$$=\begin{bmatrix} 0&0&0&0&0&0&0&\frac{10}{3} \\ 0&0&0&0&0&0&0&35 \\ 0&0&0&0&0&0&0&-\frac{10}{501} \\ 0&0&0&0&0&0&0&\frac{1}{375} \\ 0&0&0&0&0&0&0&-\frac{1}{150} \\ 0&0&0&0&0&0&0&-\frac{1}{135} \\ 0&0&0&0&0&0&0&\frac{29}{534} \\ 0&0&0&0&0&0&0&-\frac{823}{1331} \end{bmatrix}\begin{bmatrix} y_n \\ y_{n-2} \\ y_{n-3} \\ y_{n-4} \\ y_{n-5} \\ y_{n-6} \\ y_{n-7} \\ y_{n-8} \end{bmatrix} \quad (18)$$

where,

$$A^{(0)}=\begin{bmatrix} 70 & 420z+749 & 700-525 & -350 & 70 & -\frac{35}{3} & 1 & 0 \\ 720 & 4200z+7350 & 6720z-5376 & -3150 & 560 & -70 & 0 & 1 \\ 1+\frac{60}{167}z & \frac{175}{167}+\frac{300}{167}z & \frac{175}{167}+\frac{300}{167}z & \frac{-50}{167} & \frac{5}{167} & \frac{-1}{501} & 0 & 0 \\ \frac{8}{125} & \frac{12}{25}z+1 & \frac{32}{25}z & \frac{12}{25}z-1 & -\frac{8}{125} & \frac{1}{375} & 0 & 0 \\ -\frac{3}{20} & -z-\frac{23}{12} & -2z+1 & \frac{3}{2} & \frac{1}{5}z-\frac{127}{300} & -\frac{1}{60} & 0 & 0 \\ -\frac{4}{25} & -z-\frac{11}{6} & -\frac{16}{9}z+\frac{32}{17} & 1 & -\frac{4}{15} & \frac{91}{1350}-\frac{1}{45}z & 0 & 0 \\ \frac{302}{267} & \frac{597}{89}z+\frac{4225}{356} & \frac{1955}{178}z-\frac{755}{89} & -\frac{2845}{534} & 1 & \frac{-143}{1068} & \frac{1}{178}z & 0 \\ \frac{-16776}{1331} & -\frac{96660}{1331}z-\frac{167755}{1331} & -\frac{151968}{1331}+\frac{124800}{1331} & \frac{69345}{1331} & -\frac{11768}{1331} & 1 & 0 & \frac{-12}{1331}z \end{bmatrix} \quad (19)$$

$$A^{(1)}=\begin{bmatrix} 0&0&0&0&0&0&0&\frac{10}{3} \\ 0&0&0&0&0&0&0&35 \\ 0&0&0&0&0&0&0&-\frac{10}{501} \\ 0&0&0&0&0&0&0&\frac{1}{375} \\ 0&0&0&0&0&0&0&-\frac{1}{150} \\ 0&0&0&0&0&0&0&-\frac{1}{135} \\ 0&0&0&0&0&0&0&\frac{29}{534} \\ 0&0&0&0&0&0&0&-\frac{823}{1331} \end{bmatrix} \quad (20)$$

Substituting the values of $A^{(0)}$ and $A^{(1)}$ into the stability polynomial $\rho(\lambda)=\det(\lambda A^{(0)}-A^{(1)})$ gives

$$\rho(\lambda)=$$
$$-\frac{451584}{98913265}r^8z^8+\frac{6136704}{494566325}r^8z^7-\frac{13229888}{494566325}r^8z^6+\frac{56448}{98913265}r^7z^7+\frac{254016}{5556925}r^8z^5$$
$$+\frac{12096}{4087325}r^7z^6-\frac{30171456}{494566325}r^8z^4+\frac{4412352}{494566325}r^7z^5+\frac{6096384}{98913265}r^8z^3+\frac{9097536}{494566325}r^7z^4$$
$$-\frac{4402944}{98913265}r^8z^2+\frac{526848}{19782653}r^7z^3+\frac{2032128}{98913265}r^8z+\frac{2596608}{98913265}r^7z^2-\frac{451584}{98913265}r^8$$
$$+\frac{1580544}{98913265}r^7z+\frac{451584}{98913265}r^7$$

(21)

Differentiating (21) with respect to z, gives

$$-\frac{3612672}{98913265}r^8z^7+\frac{42956928}{494566325}r^8z^6-\frac{79379328}{494566325}r^8z^5+\frac{395136}{98913265}r^7z^6+\frac{254016}{1111385}r^8z^4$$
$$+\frac{72576}{4087325}r^7z^5-\frac{120685824}{494566325}r^8z^3+\frac{4412352}{98913265}r^7z^4+\frac{18289152}{98913265}r^8z^2+\frac{36390144}{494566325}r^7z^3$$
$$-\frac{8805888}{98913265}r^8z+\frac{1580544}{19782653}r^7z^2+\frac{2032128}{98913265}r^8+\frac{5193216}{98913265}r^7z-\frac{1580544}{98913265}r^7$$
$$=0 \quad (22)$$

The stability polynomial (21) and its derivative (22) are plotted in a MATLAB environment to give the absolute stability region of the BMBDF method for $k=8$ in fig. 1.

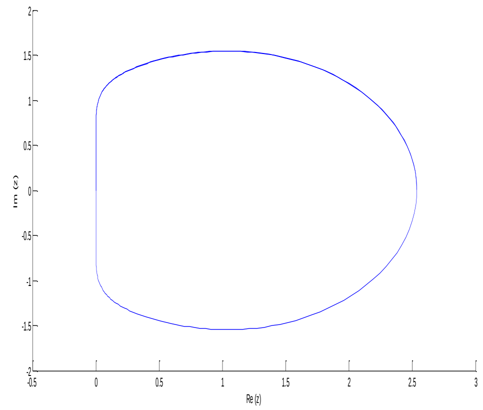

Fig. 1. Region of Absolute Stability of the BMBDF for $k=8$

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IX. NUMERICAL EXPERIMENTS

In this section, the newly constructed BMBDF for $k = 8$ was used to solve some real life systems of ordinary differential equations.

Problem 1: Moderately Stiff Linear System

$$y_1' = 998y_1 + 1998y_2$$
$$y_2' = -999y_1 - 1999y_2$$
$$y_1(0) = 1, y_2(0) = 1, t \in [0,10], h = 0.1$$

The exact solution is given by

$$y_1(t) = 4\exp(-t) - 3\exp(-1000t)$$
$$y_2(t) = -2\exp(-t) + 3\exp(-1000t)$$

[NB: This is the torsion spring oscillator with dry and viscous friction and it arises in physics]

Problem 2: Stiff Non Linear chemical reaction problem

$$y_1' = -0.1y_1 - 199.9y_2$$
$$y_2' = -200y_2$$
$$y_1(0) = 2, y_2(0) = 1, x \in [0,2], h = 0.1$$

The exact solution is given by

$$y_1(x) = \exp(-0.1x) + \exp(-200x)$$
$$y_2(x) = \exp(-200x)$$

Problem 3: Stiff Linear System

$$y_1' = -21y_1(x) + 19y_2(x) - 20y_3(x)$$
$$y_2' = 19y_1(x) - 21y_2 + 20y_x(x)$$
$$y_3' = 40y_1(x) - 40y_2(x) - 40y_3(x)$$
$$y_1(0) = 1, y_2(0) = 0, y_3(0) = -1, x \in [0,10], h = 0.1$$

The exact solution is

$$y_1(x) = \frac{1}{2}(e^{-2x} + e^{-40x}(\cos(40x) + \sin(40x))$$
$$y_2(x) = \frac{1}{2}(e^{-2x} - e^{-40x}(\cos(40x) + \sin(40x))$$
$$y_3(x) = \frac{1}{2}(2e^{-40x}(\sin(40x) - \cos(40x))$$

Problem 4: Stiff Non Linear System (Van der pol's Equations)

$$y_1' = y_2$$
$$y_2' = -y_1 + \mu y_2(1 - y_1^2)$$
$$\mu = 10, y_1(0) = 2, y_2(0) = 0, h = 0.01$$
$$t \in [0,70]$$

The *Van der Pol's Equation* is an important kind of second-order non-linear auto-oscillatory equation. It is a non-conservative oscillator with non-linear damping.

X. SOLUTION CURVES

The solution curves to problems 1 and 4 are displayed in Figs 2 & 3

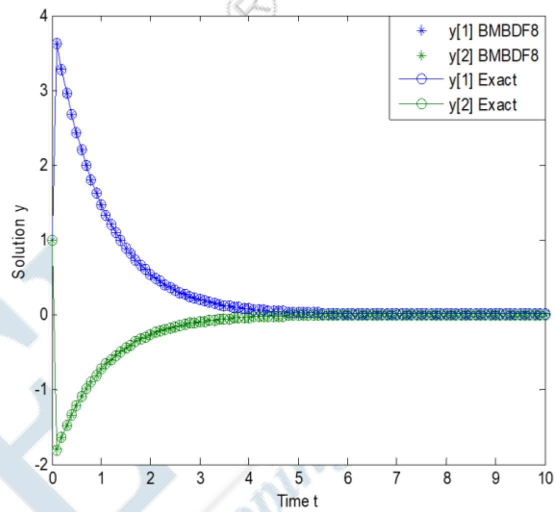

Fig. 2. Solution Curve for Problem 1Using the BMBDF k = 8

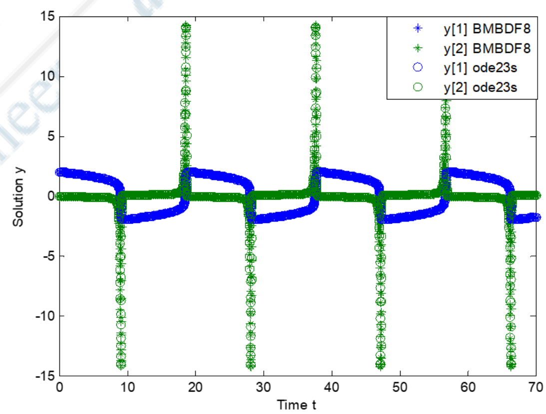

Fig. 3. Solution Curve for Problem 4 Using the BMBDF k = 8

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XI. ABSOLUTE ERRORS

The absolute errors of problems 1, 2 and 3 compared with other existing methods are shown on tables 1 – 3

Table I: Absolute Errors for Problem 1 Using Eight Step Methods

GBDF8 (Brugnano and Trigiante) BMBDF8				
x	**err 1**	**err2**	**err1**	**err2**
1.0	1.53E-06	7.63E-07	2.69E-09	1.35E-09
1.5	9.49E-07	4.75E-07	2.62E-09	1.31E-09
2.0	5.79E-07	2.89E-07	2.19E-09	1.10E-09

Table II: Absolute Errors for Problem 2 Using Eight Step Methods

GBDF8 (Brugnano and Trigiante) BMBDF8				
X	**err 1**	**err2**	**err 1**	**err2**
1.0	5.33E-08	5.78E-08	3.09E-11	9.58E-22
1.5	4.48E-09	3.18E-12	3.02E-11	6.16E-22
2.0	2.12E-09	5.62E-11	2.98E-11	5.86E-22

Table III: Absolute Errors of the first component to Problem 3 Using Eight Step Methods

GBDF8, Brugnano and Trigiante (1998) BMBDF8		
X	error	error
1.0	9.02E-07	4.43E-08
1.5	1.01E-08	2.44E-08
2.0	6.29E-06	1.20E-08

XII. DISCUSSION OF RESULTS

The solution curves in Fig. 2 and 3 shows the performance of the new BMBDF $k = 8$ on a linear and non-linear systems of ordinary differential equations when compared with the exact solution and ode23 solver. It is observed that the BMBDF gave good approximations to the stiff problems. The solution curve to problem 1 was plotted within the range of 0 < x < 10 and showed very good performance when compared with the exact solution. Similarly, the solution curve to the Van der pol equation compete exceedingly well when compared with ode23 solver.

The results of Problems 1, 2 and 3 using the BMBDF $k = 8$ were compared with the generalised backward differentiation formulae (GBDF) of Brugnano and Trigiante (1998) as shown on Tables 1 - 3. The results show that the new methods performed well with marginal absolute errors and converge much faster to the theoretical solution than that of Brugnano and Trigiante (1998).

XIII. CONCLUSION

This study has focused on the derivation of a modified linear multistep method based on the backward differentiation formulae for the solutions of stiff initial value problems of ordinary differential equations. The multistep collocation approach was used to derive the block form of the MBDF $k = 8$.

Convergence analysis of the new method was carried out and it was observed that the method was consistent and zero stable, hence convergent. Also, the region of absolute stability of the new method was plotted and showed that the method is A – stable.

The newly constructed method was used to solve real life initial value problems of ODEs and the results were seen to be efficient and accurate. The results of the study showed that the new block method compete favourably well with those of some well-known researchers.

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